## **Lecture 2. Mathematical physics equations as mathematical models**

The general question: why it can be interesting for us? We consider the general mathematical physics equations as mathematical models of physical laws.

### **2.1. Heat equation and its interpretations**

Consider the heat transfer phenomenon. Consider a long enough and thin enough body. Then the characteristics of the body depends from the unique special variable *x.* Suppose we heat the body of the mass *m* from the temperature to the temperature *u*2. It is necessary to have the heat quantity

*Q = cm* (*u*2 – *u*1)

for it, where *c* is the heat capacity of the material.

Note that the mass depends from the size of the body. Then we determine the relation between the mass and the density that is the real characteristic of the material. The density is the mass of the unit volume. If we consider one-dimensional body, then the density is the mass of the unit interval. If we consider the body with the length *X*, then the corresponding mass is

*m = ρX*,

where *ρ* is the density of the body. Therefore, the have the following formula for the heat quantity

*Q = cρ* (*u*2 – *u*1) *X.*

Let *u*1 is the temperature of the body at the time *t*, *u*2 is the temperature of the body at the time *t+*Δ*t.* Now we transform the previous formula to the equality

*Q = cρ* (*u*|*t*– *u*|*t+*Δ*t*) *X.* (2.1)

The formula (2.1) give us the value of the heat quantity under the supposition that the temperature does not change on the special interval [0,*X*]. In reality, the temperature *u* depends from *x.* Hence, this formula can be true, if the length of this interval is small enough. Suppose *X* is equal to the infinite small value *dx.* Denote by *dQ* the corresponding value of the heat quantity. Then we have

*dQ = cρ* (*u*|*t+*Δ*t* – *u*|*t*) *dX.* (2.2)

Let us consider the special interval from the coordinate *x* to the coordinate *x+*Δ*x*. For calculating the corresponding value of the heat quantity, it is necessary to integrate the equality (2.2) by *x* from the value *x* to *x+*Δ*x*. Denote the result by *Q*1. We get

 (2.3)

Now we have the question. What is the cause of the temperature change? The temperature value changes because of the heat flux. Heat flus *q* is the heat quantity that moves through the unit area during the unit time. If we consider one-dimensional body, then we consider the heat flux through the concrete point. Then we can calculate the heat quantity at the concrete point during the time *T* by the formula

*Q = qT.*

Return to the consideration of the special interval [*x*,*x+*Δ*x*]. Suppose the directions of the heat flux and the axe *x* are same. Then the difference between the heat flux at the end and the begin of this interval is

*Q =* (*q*|*x+*Δ*x* – *q*|*x*) *T.*  (2.4)

Our next problem is the calculation of the heat flux. It is obviously, the heat flux between two points of the body is proportional to the difference between its temperatures and inverse proportional to the distance between these points. Besides, we have the heat flux from the hot part of the body to its cold part. Particularly, of the temperature increase from the first point to the second point, then we have the heat flux in the inverse direction. Thus, we obtain the formula

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where *x*1 and *x*2 are the coordinates of the points, *u*1 and *u*2 are the temperature at the considered points, and *λ* is the thermal conductivity of the body, that is the property of the material. In reality, we have the continuous body, no two separated points. Then the last formula is true if we consider infinite small interval *dx.* Now we have the following equality

 (2.5)

that is called ***Fourier law***.

Put the result to the formula (2.4). We get



This formula uses the supposition that the heat flus does not changes during the time interval from 0 to *T.* In reality, this value depends from time. Then this formula is true, if we consider the infinite small time interval *dt.* Denote the corresponding value of the heat quantity by *dQ*. We have



If we would like to determine the value for the time interval [*t*,*t+*Δ*t*], it is necessary to integrate the last equality by *t* from *t* to *t+*Δ*t.* Denoting the corresponding value of theheat quantity by *Q*2. We obtain

 (2.6)

We suppose that the heat quantity changes by heat flux only. Therefore, we have the equality

*Q*1 = *Q*2.

Put here the values of the heat quantity from the equalities (2.3) and (2.6). We get

 (2.7)

The formula (2.7) characterizes the change of the heat quantity on the special interval [*x*,*x+*Δ*x*] during the time interval [*t*,*t+*Δ*t*].

Our next step is using of the ***main integral theorem***



where *y*\* is the point from [*a*,*b*]. Transform (2.7)



where *x*\*∈[*x*,*x+*Δ*x*], *t*\*∈[*t*,*t+*Δ*t*]. Divide this equality by Δ*x*Δ*t.*



After passing to the limit as Δ*x* → 0, Δ*t* → 0, we get

 (2.8)

This is the ***heat equation***. For the homogeneous body we have



It has the shorter form

*ut = a*2 *uxx* .

Consider extensions:

* Heat source



where *f* is the density of heat source.

* Convection



where *v* is the velocity.

* Heat exchange with environment



where *u\** is the temperature of the environment, *k* is the coefficient of the heat exchange.

* Multidimensional cases

For two-dimensional case we have



For three-dimensional case we have



If our body is homogeneous, then we have the equation

*ut = a*2 (*uxx* + *uyy* + *uzz*).

Denote

Δ*u* = *uxx* + *uyy* + *uzz*.

The operator Δ is called the ***Laplace operator***. Then the general heat equation has the form

*ut = a*2 Δ*u.*

Other interpretation: use the following analogy:

heat transfer diffusion electroconductivity

heat quantity mass charge

temperature concentration charge density

heat flux diffusion flux charge flux

heat conductivity diffusion coefficient charge conductivity

### **2.2. String vibrating equation**

Consider the oscillation of the long thin string in the plane. This phenomenon is characterized by the function *u = u*(*x*,*t*) that is the position of the string at the point *x* at the time *t.*

The impulse of the string is calculated by the formula

*J = mv*,

where *m* is the mass of the string, and *v* is the velocity. The velocity is the derivative of the coordinate. Then we have



Use the relation between the mass and the density. The density of the one-dimensional body is the mass of the unit interval. If we consider the string with the length *X*, then the corresponding mass is

*m = ρX*.

Now we calculate the impulse by the formula



We have an interest to the change of the impulse during the time interval from the time *t* to the time *t+*Δ*t.* Then we obtain

 (2.9)

The formula (2.9) give us the value of the impulse change under the supposition that the velocity does not change on the special interval [0,*X*]. In reality, it depends from *x.* Hence, this formula can be true, if the length of this interval is small enough. Suppose *X* is equal to the infinite small value *dx.* Denote by *dJ* the corresponding value of the impulse change. Then we have

 (2.10)

If we would like to calculate the impulse change of the special interval from the point *x* to the point *x+*Δ*x*, it is necessary to integrate the equality (2.10) by *x.* Denote the result by *J*1. We get

 (2.11)

The impulse is changes under a force. Particularly, the value of the impulse by the force *F* during the time *T* is

*J = F T.*

Now we have the tension force. This force has the direction of the tangent line to the curve that is the position to the string. We have an interest to the movement in the perpendicular direction to the axe *x.* Therefore, we have the formula

*F = τ* sin *α*,

where *k* is the tension that is the characteristic of the material, and *α* is the angle between the tangent line and the axe *x.* Let us consider the small oscillation of the string. In this situation, the sinus of the angle is approximately equal to its tangent. Now we get

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In reality, we have the movement, because there exists a difference of the tension forces at the different points of the string. Calculate the difference between the tension forces at the points *x+*Δ*x* and *x.* We determine

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Put it to the formula of the impulse. We obtain

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This formula uses the supposition that the tension force does not changes during the time interval from 0 to *T.* In reality, this value depends from time. Then this formula is true, if we consider the infinite small time interval *dt.* Denote the corresponding value of the heat quantity by *dJ*. We have

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If we would like to determine the value for the time interval [*t*,*t+*Δ*t*], it is necessary to integrate the last equality by *t* from *t* to *t+*Δ*t.* Denoting the corresponding value of theheat quantity by *J*2. We obtain

** (2.12)

We suppose that the impulse changes by tension force only. Therefore, we have the equality

*J*1 = *J*2.

Put here the values of the impulse from the equalities (2.11) and (2.12). We get

** (2.13)

The formula (2.13) gives us the balance relation of the impulse on the special interval [*x*,*x+*Δ*x*] during the time interval [*t*,*t+*Δ*t*].

Using the main integral theorem, we have

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Divide this equality by Δ*x*Δ*t*

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Passing to the limit as Δ*x* → 0, Δ*t* → 0, we get

** (2.14)

This is the ***string vibrating equation***.

If the string is homogeneous, then we have the following form of the ***string vibrating equation***

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It has the shorter form

*utt = a*2 *uxx* .

Extensions:

* Vibrating under the exterior force

*utt = a*2 *uxx* + *f*.

* Two-dimensional case (membrane vibrating equation)

*utt = a*2 (*uxx* + *uyy*)

* Three-dimensional case (wave equation).

*utt = a*2 (*uxx* + *uyy* + *uzz*)

Using Laplace operator, we have

*utt = a*2 Δ*u.*

### **2.3. Poisson and Laplace equations and their interpretation**

These systems have many interpretations.

2.3.1. Stationary heat transfer phenomenon

We now that the heat transfer in the homogeneous environment is described by the heat equation

*ut = a*2Δ*u.*

If we consider the stationary phenomenon, then the temperature does not depends from the time. Then its derivative with respect to the time is equal to zero. Then we obtain the formula

Δ*u =* 0*.* (2.15)

This equality is called the ***Lanlace equation***. If have a heat source, that is does not depend from the time, then we have the following non-homogeneous equation

Δ*u = f*, (2.16)

which is called the ***Poisson equation***.

2.3.2. Potential movement of the liquid

The movement of the liquid is characterized by the vector function velocity *v.* If we have the non-vortex flow, then there exists a scalar function *u*, which is called the potential of the velocity field, such that

*v = –*∇*u.* (2.17)

If we do not have any sources, then the velocity vector satisfies the equality

div *v =* 0*.*

Put here the value of the velocity from the formula (2.18). We get

div ∇*u =* 0*.*

Using the definition of the divergence and the gradient, we transform this equality to the Laplace equation

Δ*u =* 0*.*

2.3.3. Potential of the electric field

Consider a stationary electricity in an environment. We know Ohm law

*j = λE*,

where *j* is the currant density, *E* is the electricity field, and *λ* is the conductivity. The current density satisfies the equality

div *j =* 0,

which is the corollary of the Maxwell equations. Then we have

div *E =* 0.

The electricity potential *u* satisfies the equality

*E = –*∇*u.*

Then we can determine the electricity potential from the Laplace equation.

### **Conclusions**

* Mathematical physics equations are mathematical models of physical phenomena.
* The heat equation, string vibrating (wave) equation, and Laplace (Poisson) equation are the most important partial differential equations.
* These equations are the general subject of this course.
* Mathematical physics equations can have different physical interpretations.
* The heat equation is the mathematical model of the different transfer phenomena.
* The wave equation is the mathematical model of the different oscillation phenomena.
* The Laplace equation is the mathematical model of the different stationary phenomena.

### **Next step**

It is necessary to determine what the partial differential equations are. We will consider the classification of these equations.

### **Task 2. Partial differential equations of the first order**

Consider the first order partial differential equation

, 0<*x*<*L*, 0<*y*<*M*, (1)

where *a*, *L*, *M* are given constants. If *a*>0, then we can have the boundary conditions

 (2)

or

 (3)

If *a*<0, then we can have the boundary conditions

 (4)

or

, (5)

where *ϕ* and *ψ* are given functions.

It is necessary to find the solution of the given problem, using characteristic method. Check that the result is, in reality, the solution of the problem.

Table of parameters

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **variant** | ***a*** | **boundary**  **conditions** | ***L*** | ***M*** | *ϕ*(*y*) | *ψ*(*x*) |
| 1 | 2 | (2) | 1 | 2 | sin π*y* | sin 2π*x* |
| 2 | -2 | (4) | 2 | 1 | cos 2π*y* | cos π*x* |
| 3 | 1/2 | (3) | π/2 | π | -sin *y* | -sin 2*x* |
| 4 | -1/2 | (5) | π | π/2 | -cos 2*y* | -cos *x* |
| 5 | 3 | (2) | π | π/2 | sin 2*x* | sin *y* |
| 6 | -3 | (4) | π/2 | π | cos *y* | cos 2*x* |
| 7 | 1/3 | (3) | 2 | 1 | -sin 2π*y* | -sin π*x* |
| 8 | -1/3 | (5) | 1 | 2 | -cos π*y* | -cos 2π*x* |